Ion-Acoustic Waves in Unmagnitized Collisionless Weakly Relativistic Plasma using Time-Fractional KdV Equation

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Abstract

The reductive perturbation method has been employed to derive the Korteweg–de Vries (KdV) equation for small but finite amplitude electrostatic ion-acoustic waves in unmagnitized collisionless weakly relativistic warm plasma. The Lagrangian of the time fractional KdV equation is used in similar form to the Lagrangian of the regular KdV equation. The variation of the functional of this Lagrangian leads to the Euler-Lagrange equation that leads to the time fractional KdV equation. The Riemann-Liouvulle definition of the fractional derivative is used to describe the time fractional operator in the fractional KdV equation. The variational-iteration method given by He is used to solve the derived time fractional KdV equation. The calculations of the solution with initial condition $A_0\,\mathrm{sech}(cx)^2$ are carried out. The result of the present investigation may be applicable to some plasma environments, such as ionosphere.

Keywords **:** Ion-acoustic waves; Euler-Lagrange equation, Riemann-Liouvulle fractional derivative, fractional KdV equation, He's variational-iteration method.

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1 Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena. Because most classical processes observed in the

physical world are nonconservative, it is important to be able to apply the power of variational methods to such cases. A method used a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative. It was shown that such fractional derivatives in the Lagrangian describe nonconservative forces [1, 2]. Further study of the fractional Euler-Lagrange can be found in the work of Agrawal [3, 4], Baleanu and coworkers [5, 6] and Tarasov and Zaslavsky [7, 8]. During the last decades, Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Some of the areas where Fractional Calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory [9].

The propagation of solitary waves is important as it describes characteristic nature of the interaction of the waves and the plasmas. In the case where the velocity of particles is much smaller than that of light, ion-acoustic waves present the non-relativistic behaviors, but in the case where the velocity of particles approaches that of light, the relativistic effect becomes dominant [10]. Actually high speed and energetic streaming ions with the energy from 0.1 to 100 MeV are frequently observed in solar atmosphere and interplanetary space. Nevertheless, relativistic ion-acoustic waves have not been well investigated. When we assume that the ion energy depends only on the kinetic energy, such plasma ions have to attain very high velocity of relativistic order. Thus, by considering the weakly relativistic effect where the ion velocity is about $\frac{1}{10}$ of the velocity of light, we can describe the relativistic motion of such ions in the study of nonlinear interaction of the waves and the plasmas [11]. It appears that the weakly relativistic and ion temperature effects play an important role to energetic ion-acoustic waves propagating in interplanetary space [12]. Washimi and Taniti [13] were the first to use reductive perturbation method to study the propagation of a slow modulation of a quasimonochromatic waves through plasma. And then the attention has been focused by many authors [14–15].

To the author's knowledge, the problem of time fractional KdV equation in weakly relativistic plasma has not been addressed in the literature before. So, our motive here is to study the effects of time fractional parameter on the electrostatic structures for a system of collisionless plasma consisting of a mixture of warm ion-fluid and isothermal electrons with ion flow velocity has a weakly relativistic effect. We expect that the inclusion of time fractional parameter and a weakly relativistic effect will change the properties as well

as the regime of existence of solitons.

Several methods have been used to solve fractional differential equations such as: the Laplace transform method, the Fourier transform method, the iteration method and the operational method [16]. Recently, there are some papers deal with the existence and multiplicity of solution of nonlinear fractional differential equation by the use of techniques of nonlinear analysis [17-18]. In this paper, the resultant fractional KdV equation will be solved using a variational-iteration method (VIM) firstly used by He [19].

This paper is organized as follows: Section 2 is devoted to describe the formulation of the time-fractional KdV (FKdV) equation using the variational Euler-Lagrange method. In section 3, Variational-Iteration Method (VIM) is discussed. The resultant time-FKdV equation is solved approximately using VIM. Section 5 contains the results of calculations and discussion of these results.

2 Basic equations and KdV equation

Consider collisionless ionization-free unmagnetized plasma consisting of a mixture of warm ions-fluid and isothermal electrons. Assume that the ions flow velocity has a weak relativistic effect, and therefore there exist streaming ions in an equilibrium state when sufficiently small- but finite-amplitude waves propagate one-dimensionally. Such a system is governed by the following normalized equations [12]:

$$\frac{\partial}{\partial t}n(x,t)+\frac{\partial}{\partial x}[n(x,t)u(x,t)] = 0, \quad (1a)$$

$$[\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x}][\gamma(x,t)u(x,t)]+\frac{\sigma}{n(x,t)}\frac{\partial}{\partial x}p(x,t)+\frac{\partial}{\partial x}\phi(x,t) = 0, \quad (1b)$$

$$[\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x}]p(x,t)+3p(x,t)\frac{\partial}{\partial x}[\gamma u(x,t)] = 0, \quad (1c)$$

$$\frac{\partial^2}{\partial x^2}\phi(x,t)+n(x,t)-n_e(x,t) = 0. \quad (1d)$$

The electrons temperature T_e is much larger than the ions temperature T_i and in this case, for simplicity, one can neglect the inertia of the electrons relative to that of the ions, i.e. the high-frequency plasma oscillations are neglected. Since we are interested in the regime of density and velocity fluctuations near the ion plasma frequency, so the isothermal electrons given by:

$$n_e(x,t)=\exp[\phi(x,t)], \quad (1e)$$

for weakly relativistic effects, the relativistic factor $\gamma(x,t) = [1-u(x,t)^2/c_0^2]^{-\frac{1}{2}}$ is approximated by

$$\gamma(x,t) \approx 1 + u(x,t)^2/(2c_0^2), \tag{1f}$$

where c is the velocity of light. In equations (1) $n(x,t)$ and $n_e(x,t)$ are the densities of ions and electrons, respectively, $u(x,t)$ is the ions flow velocity, $p(x,t)$ is the ions pressure, $\phi(x,t)$ is the electric potential, x is the space co-ordinate and t is the time variable. $\sigma = T_i/T_e << 1$ is the ratio of the ions temperature to the electrons temperature. All these quantities are dimensionless, being normalized in terms of the following characteristic quantities: $n(x,t)$ and $n_e(x,t)$ are normalized by the unperturbed electrons density n_0, $u(x,t)$ and c are normalized by the sound velocity $(k_B T_e/m_i)^{1/2}$, $p(x,t)$ and $\phi(x,t)$ are normalized by $n_0 k_B T_i$ and $k_B T_e/e$, respectively and t and x are normalized by the inverse of the plasma frequency $\omega_{pi}^{-1} = (4\pi e^2 n_0/m_i)^{-1/2}$ and the electron Debye length $\lambda_D = (k_B T_e/4\pi e^2 n_0)^{1/2}$, respectively. k_B is the Boltzmann's constant, e is the elecron charge and m_i is the mass of plasma ion.

According to the general method of reductive perturbation theory, we introduce the stretched variables

$$\tau = \epsilon^{\frac{3}{2}} t, \;\; \xi = \epsilon^{\frac{1}{2}}(x - \lambda t), \tag{2}$$

where λ is the the phase velocity. All the physical quantities appeared in (1) are expanded as power series in terms of the amplitude of the perturbation ϵ about the equilibrium values as

$$\begin{aligned}
n(\xi,\tau) &= 1 + \epsilon n_1(\xi,\tau) + \epsilon^2 n_2(\xi,\tau) + ..., && \text{(3a)}\\
u(\xi,\tau) &= u_0 + \epsilon u_1(\xi,\tau) + \epsilon^2 u_2(\xi,\tau) + ..., && \text{(3b)}\\
p(\xi,\tau) &= 1 + \epsilon\; p_1(\xi,\tau) + \epsilon^2 p_2(\xi,\tau) + ..., && \text{(3c)}\\
\phi(\xi,\tau) &= \epsilon \phi_1(\xi,\tau) + \epsilon^2 \phi_2(\xi,\tau) + ..., && \text{(3d)}
\end{aligned}$$

with the boundary conditions that as $|\xi| \to \infty$, $n = n_e = p = 1$, $u = u_0$, $\phi = 0$.

Substituting (2) and (3) into the system of equations (1) and equating coefficients of like powers of ϵ. Then, from the lowest, second-order equations in ϵ and by eliminating the second order perturbed quantities $n_2(\xi,\tau)$, $u_2(\xi,\tau)$, $p_2(\xi,\tau)$ and $\phi_2(\xi,\tau)$, we obtain the following KdV equation for the first-order perturbed potential:

$$\frac{\partial}{\partial\tau}\phi_1(\xi,\tau) + A\;\phi_1(\xi,\tau)\frac{\partial}{\partial\xi}\phi_1(\xi,\tau) + B\;\frac{\partial^3}{\partial\xi^3}\phi_1(\xi,\tau) = 0, \tag{4a}$$

where

$$A = \lambda - \left(\lambda^2 - 3\sigma\right)\frac{\gamma_2}{\gamma_1} + \frac{3\sigma}{2\lambda}(3\gamma_1 - 1), \qquad B = \frac{(\lambda^2 - 3\sigma)}{2\lambda}, \tag{4b}$$

with the transendental equation for λ as

$$1 - \left(\lambda^2 - 3\sigma\right)\gamma_1 = 0 \implies \lambda = \pm\sqrt{\frac{3\sigma\gamma_1 + 1}{\gamma_1}}, \tag{4c}$$

and

$$\gamma_1 = 1 + \frac{3}{2}R^2, \qquad \gamma_2 = \frac{3}{2}\frac{R}{c_0}, \qquad R = \frac{u_0}{c_0}. \tag{4d}$$

In equation (4a), $\phi_1(\xi,\ \tau)$ is a field variable, ξ is a space coordinate in the propagation direction of the field and $\tau \in T\ (= [0, T_0])$ is the time.f

The resultant KdV equation (4a) can be converted into time-fractional KdV equation as follows: Using a potential function $V(\xi,\ \tau)$ where $\phi_1(\xi,\ \tau) = V_\xi(\xi,\ \tau) = \Phi(\xi,\ \tau)$ gives the potential equation of the regular KdV equation (1) in the form

$$V_{\xi\tau}(\xi,\ \tau) + A\ V_\xi(\xi,\ \tau)v_{\xi\xi}(\xi,\ \tau) + B\ V_{\xi\xi\xi\xi}(\xi,\ \tau) = 0, \tag{5}$$

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Lagrangian of this regular KdV equation (4a) can be defined using the semi-inverse method [20, 21] as follows.

The functional of the potential equation (5) can be represented by

$$J(V) = \int_R d\xi \int_T d\tau \{V(\xi,\tau)[c_1 V_{\xi\tau}(\xi,\tau) + c_2 A V_\xi(\xi,\tau)v_{\xi\xi}(\xi,\tau) + c_3 B V_{\xi\xi\xi\xi}(\xi,\tau)]\}, \tag{6}$$

where c_1, c_2 and c_3 are constants to be determined. Integrating by parts and taking $V_\tau|_R = V_\xi|_R = V_\xi|_T = 0$ lead to

$$J(V) = \int_R d\xi \int_T d\tau \{V(\xi,\tau)[-c_1 V_\xi(\xi,\tau)V_\tau(\xi,\tau) - \frac{1}{2}c_2 A V_\xi^3(\xi,\tau) + c_3 B V_{\xi\xi}^2(\xi,\tau)]\}. \tag{7}$$

The unknown constants c_i $(i = 1,\ 2,\ 3)$ can be determined by taking the variation of the functional (7) to make it optimal. Taking the variation of this functional, integrating each term by parts and making the variation optimum give the following relation

$$2c_1 V_{\xi\tau}(\xi,\tau) + 3c_2 A V_\xi(\xi,\tau)V_{\xi\xi}(\xi,\tau) + 2c_3 B V_{\xi\xi\xi\xi}(\xi,\tau) = 0. \tag{8}$$

As this equation must be equal to equation (5), the unknown constants are given as

$$c_1 = 1/2,\ c_2 = 1/3 \text{ and } c_3 = 1/2. \tag{9}$$

Therefore, the functional given by (7) gives the Lagrangian of the regular KdV equation as

$$L(V_\tau,\ V_\xi, V_{\xi\xi}) = -\frac{1}{2}V_\xi(\xi,\tau)V_\tau(\xi,\tau) - \frac{1}{6}AV_\xi^3(\xi,\tau) + \frac{1}{2}BV_{\xi\xi}^2(\xi,\tau). \tag{10}$$

Similar to this form, the Lagrangian of the time-fractional version of the KdV equation can be written in the form

$$\begin{aligned} F({}_0D_\tau^\alpha V,\ V_\xi, V_{\xi\xi}) &= -\frac{1}{2}[{}_0D_\tau^\alpha V(\xi,\tau)]V_\xi(\xi,\tau) - \frac{1}{6}AV_\xi^3(\xi,\tau) + \frac{1}{2}BV_{\xi\xi}^2(\xi,\tau), \\ 0 &\leq \alpha < 1, \end{aligned} \tag{11}$$

where the fractional derivative is represented, using the left Riemann-Liouville fractional derivative definition as [16]

$$\begin{aligned} {}_aD_t^\alpha f(t) &= \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_a^t d\tau(t-\tau)^{k-\alpha-1}f(\tau)], \\ k-1 &\leq \alpha \leq k,\ t \in [a,b]. \end{aligned} \tag{12}$$

The functional of the time-FKdV equation can be represented in the form

$$J(V) = \int\limits_R d\xi \int\limits_T d\tau F({}_0D_\tau^\alpha V,\ V_\xi, V_{\xi\xi}), \tag{13}$$

where the time-fractional Lagrangian $F({}_0D_\tau^\alpha V,\ V_\xi, V_{\xi\xi})$ is defined by (11).

Following Agrawal's method [3, 4], the variation of functional (13) with respect to $V(\xi,\tau)$ leads to

$$\delta J(V) = \int\limits_R d\xi \int\limits_T d\tau\{\frac{\partial F}{\partial_0 D_\tau^\alpha V}\delta_0 D_\tau^\alpha V + \frac{\partial F}{\partial V_\xi}\delta V_\xi + \frac{\partial F}{\partial V_{\xi\xi}}\delta V_{\xi\xi}\}. \tag{14}$$

The formula for fractional integration by parts reads [3, 16]

$$\int_a^b dt\ f(t)\ {}_aD_t^\alpha g(t) = \int_a^b dt\ g(t)\ {}_tD_b^\alpha f(t), \qquad f(t),\, g(t)\ \in [a,\ b], \tag{15}$$

where ${}_tD_b^\alpha$, the right Riemann-Liouville fractional derivative, is defined by [16]

$$\begin{aligned} {}_tD_b^\alpha f(t) &= \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_t^b d\tau(\tau-t)^{k-\alpha-1}f(\tau)], \\ k-1 &\leq \alpha \leq 1,\ t \in [a,b]. \end{aligned} \tag{16}$$

Integrating the right-hand side of (14) by parts using formula (15) leads to

$$\delta J(V) = \int\limits_R d\xi \int\limits_T d\tau [{}_\tau D_{T_0}^\alpha (\frac{\partial F}{\partial_0 D_\tau^\alpha V}) - \frac{\partial}{\partial \xi}(\frac{\partial F}{\partial V_\xi}) + \frac{\partial^2}{\partial \xi^2}(\frac{\partial F}{\partial V_{\xi\xi}})]\delta V, \tag{17}$$

where it is assumed that $\delta V|_T = \delta V|_R = \delta V_\xi|_R = 0$.

Optimizing this variation of the functional $J(V)$, i. e; $\delta J(V) = 0$, gives the Euler-Lagrange equation for the time-FKdV equation in the form

$${}_\tau D_{T_0}^\alpha (\frac{\partial F}{\partial_0 D_\tau^\alpha V}) - \frac{\partial}{\partial \xi}(\frac{\partial F}{\partial V_\xi}) + \frac{\partial^2}{\partial \xi^2}(\frac{\partial F}{\partial V_{\xi\xi}}) = 0. \tag{18}$$

Substituting the Lagrangian of the time-FKdV equation (11) into this Euler-Lagrange formula (18) gives

$$-\frac{1}{2}\,{}_\tau D_{T_0}^\alpha V_\xi(\xi,\tau) + \frac{1}{2}\,{}_0 D_\tau^\alpha V_\xi(\xi,\tau) + AV_\xi(\xi,\tau)V_{\xi\xi}(\xi,\tau) + BV_{\xi\xi\xi\xi}(\xi,\tau) = 0. \tag{19}$$

Substituting for the potential function, $V_\xi(\xi,\tau) = \phi_1(\xi,\tau) = \Phi(\xi,\tau)$, gives the time-FKdV equation for the state function $\Phi(\xi,\tau)$ in the form

$$\frac{1}{2}[{}_0 D_\tau^\alpha \Phi(\xi,\tau) -{}_\tau D_{T_0}^\alpha \Phi(\xi,\tau)] + A\ \Phi(\xi,\tau)\ \Phi_\xi(\xi,\tau) + B\ \Phi_{\xi\xi\xi}(\xi,\tau) = 0, \tag{20}$$

where the fractional derivatives ${}_0 D_\tau^\alpha$ and ${}_\tau D_{T_0}^\alpha$ are, respectively the left and right Riemann-Liouville fractional derivatives and are defined by (12) and (16).

The time-FKdV equation represented in (20) can be rewritten by the formula

$$\frac{1}{2}\,{}_0^R D_\tau^\alpha \Phi(\xi,\tau) + A\ \Phi(\xi,\tau)\ \Phi_\xi(\xi,\tau) + B\ \Phi_{\xi\xi\xi}(\xi,\tau) = 0, \tag{21}$$

where the fractional operator ${}_0^R D_\tau^\alpha$ is called Riesz fractional derivative and can be represented by [4, 16]

$$\begin{aligned} {}_0^R D_t^\alpha f(t) &= \frac{1}{2}[{}_0 D_t^\alpha f(t) + \ (-1)^k {}_t D_{T_0}^\alpha f(t)] \\ &= \frac{1}{2}\frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}[\int_a^t d\tau |t-\tau|^{k-\alpha-1} f(\tau)], \\ k-1 &\leq \alpha \leq 1,\ t \in [a,b]. \end{aligned} \tag{22}$$

The nonlinear fractional differential equations have been solved using different techniques [16-20]. In this paper, a variational-iteration method (VIM) [21] has been used to solve the time-FKdV equation that formulated using Euler-Lagrange variational technique.

3 Variational-iteration method

Variational-iteration method (VIM) [21] has been used successfully to solve different types of integer nonlinear differential equations [22, 23]. Also, VIM is used to solve linear and nonlinear fractional differential equations [24, 25]. This VIM has been used in this paper to solve the formulated time-FKdV equation.

A general Lagrange multiplier method is constructed to solve non-linear problems, which was first proposed to solve problems in quantum mechanics [21]. The VIM is a modification of this Lagrange multiplier method [22]. The basic features of the VIM are as follows. The solution of a linear mathematical problem or the initial (boundary) condition of the nonlinear problem is used as initial approximation or trail function. A more highly precise approximation can be obtained using iteration correction functional. Considering a nonlinear partial differential equation consists of a linear part $\hat{L}U(x,t)$, nonlinear part $\hat{N}U(x,t)$ and a free term $f(x,t)$ represented as

$$\hat{L}U(x,t) + \hat{N}U(x,t) = f(x,t), \tag{23}$$

where $\hat{L}$ is the linear operator and $\hat{N}$ is the nonlinear operator. According to the VIM, the $(n+1)\underline{\text{th}}$ approximation solution of (23) can be given by the iteration correction functional as [24, 25]

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t d\tau\lambda(\tau)[\hat{L}U_n(x,\tau) + \hat{N}\tilde{U}_n(x,\tau) - f(x,\tau)], \quad n \geq 0, \tag{24}$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\tilde{U}_n(x,\tau)$ is considered as a restricted variation function, i. e; $\delta\tilde{U}_n(x,\tau) = 0$. Extreme the variation of the correction functional (24) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration can be used as the solution of the linear part of (23) or the initial value $U(x,0)$. As n tends to infinity, the iteration leads to the exact solution of (23), i. e;

$$U(x,t) = \lim_{n\to\infty} U_n(x,t). \tag{25}$$

For linear problems, the exact solution can be given using this method in only one step where its Lagrangian multiplier can be exactly identified.

4 Time-fractional KdV equation solution

The time-FKdV equation represented by (21) can be solved using the VIM by the iteration correction functional (24) as follows:

Affecting from left by the fractional operator on (21) leads to

$$\begin{aligned}\frac{\partial}{\partial\tau}\Phi(\xi,\tau) &= {}_0^R D_\tau{}^{\alpha-1}\Phi(\xi,\tau)|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)}\\ &\quad - {}_0^R D_\tau{}^{1-\alpha}[A\ \Phi(\xi,\tau)\frac{\partial}{\partial\xi}\Phi(\xi,\tau) + B\ \frac{\partial^3}{\partial\xi^3}\Phi(\xi,\tau)],\\ 0 &\leq \alpha\leq 1,\ \tau\in[0,T_0],\end{aligned} \tag{26}$$

where the following fractional derivative property is used [16]

$$\begin{aligned}{}_a^R D_b{}^\alpha[\ {}_a^R D_b{}^\beta f(t)] &= {}_a^R D_b{}^{\alpha+\beta} f(t) - \sum_{j=1}^{k} {}_a^R D_b{}^{\beta-j} f(t)|_{t=a}\ \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)},\\ k-1 &\leq \beta<k.\end{aligned} \tag{27}$$

As $\alpha<1$, the Riesz fractional derivative ${}_0^R D_\tau{}^{\alpha-1}$ is considered as Riesz fractional integral ${}_0^R I_\tau^{1-\alpha}$ that is defined by [16]

$$\begin{aligned}{}_0^R I_\tau{}^\alpha f(t) &= \frac{1}{2}[{}_0 I_\tau{}^\alpha f(t)\ +\ {}_\tau I_b{}^\alpha f(t)]\\ &= \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_a^b d\tau|t-\tau|^{\alpha-1}f(\tau),\ \alpha>0.\end{aligned} \tag{28}$$

where ${}_0 I_\tau{}^\alpha f(t)$ and ${}_\tau I_b{}^\alpha f(t)$ are the left and right Riemann-Liouvulle fractional integrals, respectively [16].

The iterative correction functional of equation (26) is given as

$$\begin{aligned}\Phi_{n+1}(\xi,\tau) &= \Phi_n(\xi,\tau) + \int_0^\tau d\tau'\lambda(\tau')\{\frac{\partial}{\partial\tau'}\Phi_n(\xi,\tau')\\ &\quad - {}_0^R I_{\tau'}^{1-\alpha}\Phi_n(\xi,\tau')|_{\tau'=0}\frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)}\\ &\quad + {}_0^R D_{\tau'}{}^{1-\alpha}[A\tilde{\Phi}_n(\xi,\tau')\frac{\partial}{\partial\xi}\tilde{\Phi}_n(\xi,\tau') + B\ \frac{\partial^3}{\partial\xi^3}\tilde{\Phi}_n(\xi,\tau')]\},\end{aligned} \tag{29}$$

where $n\geq 0$ and the function $\tilde{\Phi}_n(\xi,\tau)$ is considered as a restricted variation function, i. e; $\delta\tilde{\Phi}_n(\xi,\tau)=0$. The extreme of the variation of (29) using the

restricted variation function leads to

$$\begin{aligned}\delta\Phi_{n+1}(\xi,\tau) &= \delta\Phi_n(\xi,\tau) + \int_0^\tau d\tau'\lambda(\tau')\ \delta\frac{\partial}{\partial\tau'}\Phi_n(\xi,\tau') \\ &= \delta\Phi_n(\xi,\tau) + \lambda(\tau)\ \delta\Phi_n(\xi,\tau) - \int_0^\tau d\tau'\frac{\partial}{\partial\tau'}\lambda(\tau')\ \delta\Phi_n(\xi,\tau') = 0.\end{aligned}$$

This relation leads to the stationary conditions $1+\lambda(\tau) = 0$ and $\frac{\partial}{\partial\tau'}\lambda(\tau') = 0$, which lead to the Lagrangian multiplier as $\lambda(\tau') = -1$. Therefore, the correction functional (29) is given by the form

$$\begin{aligned}\Phi_{n+1}(\xi,\tau) &= \Phi_n(\xi,\tau) - \int_0^\tau d\tau'\{\frac{\partial}{\partial\tau'}\Phi_n(\xi,\tau') \\ &- {}_0^R I_{\tau'}^{1-\alpha}\Phi_n(\xi,\tau')|_{\tau'=0}\frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^R D_{\tau'}^{1-\alpha}[A\ \Phi_n(\xi,\tau')\frac{\partial}{\partial\xi}\Phi_n(\xi,\tau') + B\ \frac{\partial^3}{\partial\xi^3}\Phi_n(\xi,\tau')]\},\end{aligned} \tag{30}$$

where $n \geq 0$.

In Physics, if τ denotes the time-variable, the right Riemann-Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development [3]. Therefore, the right-derivative is used equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

$$\Phi_0(\xi,\tau) = \Phi(\xi,0) = A_0 \sec\mathrm{h}(c\xi)^2, \tag{31}$$

where $A_0 = \frac{3v}{A}$ and $c = \frac{1}{2}\sqrt{\frac{v}{B}}$ are constants and v is the travelling wave velocity.

Substituting this zero order approximation into (30) and using the definition of the fractional derivative (22) lead to the first order approximation as

$$\begin{aligned}\Phi_1(\xi,\tau) &= A_0 \sec\mathrm{h}(c\xi)^2 \\ &+2A_0 c\ \sinh(c\xi)\ \sec\mathrm{h}(c\xi)^3 \\ &*[4c^2 B + (A_0 A - 12c^2 B)\sec\mathrm{h}(c\xi)^2]\frac{\tau^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{32}$$

Substituting this equation into (30), using the definition (22) and the Maple package lead to the second order approximation in the form

$$
\begin{aligned}
\Phi_2(\xi,\tau) \;=\;& A_0 \sec\mathrm{h}(c\xi)^2 \\
& +2A_0 c\ \sinh(c\xi)\ \sec\mathrm{h}(c\xi)^3 \\
& *[4c^2 B + (A_0 A - 12c^2 B)\sec\mathrm{h}(c\xi)^2]\frac{\tau^\alpha}{\Gamma(\alpha+1)} \\
& +2A_0 c^2 \sec\mathrm{h}(c\xi)^2 \\
& *[32c^4 B^2 + 16c^2 B(5A_0 A - 63c^2 B)\sec\mathrm{h}(c\xi)^2 \\
& +2(3A_0^2 A^2 - 176A_0 c^2 AB + 1680c^4 B^2)\sec\mathrm{h}(c\xi)^4 \\
& -7(A_0^2 A^2 - 42A_0 c^2 AB + 360c^4 B^2)\sec\mathrm{h}(c\xi)^6]\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \\
& +4A_0^2 c^3 \sinh(c\xi)\ \sec\mathrm{h}(c\xi)^5 \\
& *[32c^4 B^2 + 24c^2 B(A_0 A - 14c^2 B)\sec\mathrm{h}(c\xi)^2 \\
& +4(A_0^2 A^2 - 32A_0 c^2 AB + 240c^4 B^2)\sec\mathrm{h}(c\xi)^4 \\
& -5(A_0^2 A^2 - 24A_0 c^2 AB + 144c^4 B^2)\sec\mathrm{h}(c\xi)^6] \\
& *\frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}.
\end{aligned} \tag{33}
$$

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.

5 Results and discussion

Numerical studies have been made for a small amplitude ion-acoustic waves in an unmagnetized collisionless plasma consisting of a mixture of a weak relativistic warm ion-fluid and isothermal electrons. We have derived the Korteweg-de Vries equation by using the reductive perturbation method [13]. The time-FKdV equation is derived from the Eular-Lagrangian using Agrawal's method [4]. The Riemann-Liouvulle fractional derivative is used to describe the time fractional operator in the FKdV equation. He's variational-iteration method [21-23] is used to solve the derived time-FKdV equation.

However, since one of our motivations was to study effects of a relativistic factor $R = u_0/c_0$ and time fractional order α on the existence of solitary waves. The present system admits a solitary wave solution for any order approximation. In Fig.(1), a profile of the bell-shaped solitary pulse is obtained. Figure (2) shows that both the amplitude and the width of the

solitary wave increases with the relativistic factor R. Also, the increasing of the time fractional order α decreases the soliton amplitude as shown in Fig (3).

In summery, it has been found that amplitude and width of the ion-acoustic waves as well as parametric regime where the solitons can exist is sensitive to the relativistic factor R. Moreover, the time fractional order α plays a role of higher order perturbation theory in varying the soliton amplitude. The application of our model might be particularly interesting in some plasma environments, such as ionosphere.

References

[1] Riewe, F., Nonconservative Lagrangian and Hamiltonian mechanics, Physical Review E 53 (1996) 1890.

[2] Riewe, F., Mechanics with fractional derivatives, Physical Review E 55 (1997) 3581.

[3] Agrawal, O. P., Formulation of Euler-Lagrange equations for fractional variational problems, J. Mathematical Analysis and Applications 272 (2002) 368.

[4] Agrawal, O. P., Fractional variational calculus in terms of Riesz fractional derivatives J. Physics A: Mathematical and Theoretical 40 (2007) 6287.

[5] Baleanu, D. and Avkar, T., Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, Nuovo Cimento B 119 (2004) 73-79.

[6] Muslih, S. I., Baleanu, D. and Rabei, E., Hamiltonian formulation of classical fields within Riemann-Liouville fractional derivatives, Physica Scripta 73 (2006) 436-438.

[7] Tarasov, V. E. and Zaslavsky, G. M., Fractional Ginzburg-Landau equation for fractal media, Physica A: Statistical Mechanics and Its Applications 354 (2005) 249-261.

[8] Tarasov, V. E. and Zaslavsky, G. M., Nonholonomic constraints with fractional derivatives, J. Physics A: Mathematical and General 39 (2006) 9797-9815.

[9] Sabatier, J., Agrawal, O. P. and Tenreiro Machado, J. A. (editors), Advances in Fractional Calculus, (Springer, Dordrecht, The Netherlands, 2007).

[10] Das, G. C. and Paul, S. N., Phys. Fluids 28 (1985) 823.

[11] Nejoh, Y., The effect of the ion temperature on the ion acoustic solitary waves in a collisionless relativistic plasma, J. Plasma Physics 37(3) (1987) 487-495.

[12] El-Labany, S. K., Contribution of higher-order nonlinearity to nonlinear ion-acoustic waves in a weakly relativistic warm plasma. Part 1. Isothermal case, J. Plasma Physics 50 (1993) 495.

[13] Washimi, H. and Taniuti, T., Propagation of Ion-Acoustic Solitary Waves of Small Amplitude, Physical Review Letters 17 (1966) 996-998.

[14] Taniuti, T. and Wei, C-C., Reductive Perturbation Method in Nonlinear Wave Propagation. I, J. Physical Society Japan 24 (1968) 941-946.

[15] El-Shewy, E. K., Zahran, M. A., Schoepf, K., Elwakil, S. A., Contribution of higher order dispersion to nonlinear dust-acoustic solitary waves in dusty plasma with different sized dust grains and nonthermal ions, Physica Scripta 78 (2008) 025501.

[16] Podlubny, I., Fractional Differential Equations, (Academic Press, San Diego, 1999).

[17] Babakhani, A. and Gejji, V. D., Existence of positive solutions of nonlinear fractional differential equations, J. Mathematical Analysis and Applications 278 (2003) 434–442.

[18] He, J.-H., Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering 167 (1998) 57-68.

[19] He, J.-H., A new approach to nonlinear partial differential equations, Communication Nonlinear Science and Numerical Simulation 2(4) (1997) 230-235.

[20] He, J.-H., Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbo-machinery aerodynamics, Int. J. Turbo Jet-Engines 14(1) (1997) 23-28.

[21] He, J.-H., Variational principles foe some nonlinear partial differential equations with variable coefficients, Chaos, Solitons and Fractals 19 (2004) 847-851.

[22] He, J.-H. and Wu, X.-H., Construction of solitary solution and compacton-like solution by variational iteration method, Chaos, Solitons and Fractals 29 (2006) 108-113.

[23] Abulwafa, E. M., Abdou M. A., Mahmoud A. A., The Variational-Iteration Method to Solve the Nonlinear Boltzmann Equation, Zeitschrift für Naturforschung A 63a (2008) 131-139.

[24] Molliq R, Y., Noorani, M. S. M. and Hashim, I., Variational iteration method for fractional heat- and wave-like equations, Nonlinear Analysis: Real World Applications 10 (2009) 1854-1869.

[25] Sweilam, N. H., Khader, M. M. and Al-Bar, R. F., Numerical studies for a multi-order fractional differential equation, Physics Letters A 371 (2007) 26–33.

Figure Captions

Fig (1): The electrostatic potential, $\Phi(\xi, \tau)$, vs ξ and τ, for $v = 0.04$, $\sigma = 0.03$, $\alpha = 0.2$ and $R = 0.03$.

Fig (2): The electrostatic potential, $\Phi(\xi, \tau)$, vs ξ at $\tau = 5$, $v = 0.04$, $\sigma = 0.03$ and $\alpha = 0.2$ for different values of R.

Fig (3): The amplitude of electrostatic potential, $\Phi(0, \tau)$, vs α at $v = 0.04$, $\sigma = 0.03$ and $R = 0.03$ for different values of τ.